Eigenanalysis of Multibody Systems

Joe Dinius
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Abstract

Eigenanalysis is integral in the study of multibody system dynamics. Understanding the linearized behavior, and the oscillatory modes and frequencies, of a nonlinear system with a large number of degrees of freedom can lend insight to the more complicated nonlinear dynamics. This paper explores the use of eigenanalysis to evaluate the natural frequencies of a few multibody systems with varying degrees of freedom. In particular, this paper is an investigation into the proper form of matrices to use in the construction of the linearized equations of motion.

Introduction and Theory

Consider the equations of motion for an arbitrary multibody system

$$\begin{pmatrix} \mathbf{M} & -\mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{v}} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \gamma \end{pmatrix}.$$

This equation contains all of the necessary information to simulate any multibody system with mass matrix $\mathbf{M}$, constraint Jacobian $\mathbf{D}$ that is subjected to the applied force vector $\mathbf{g}$. For most systems, the masses and moments-of-inertia that make up $\mathbf{M}$ are typically constant. However, in the case of many joints, the constraints between coordinates can be nonlinear (e.g. a spherical-spherical joint). In the case of nonlinear constraints, the above problem can be easy to solve numerically, but difficult to interpret analytically. Here is where eigenanalysis becomes useful.

For most nonlinear dynamical systems, the linearized equations of motion for these systems describes the full system dynamics very well near *fixed points*. In fact, these linearized equations of motion can give very good information about a system's behavior far away from a fixed point, depending on the system. Fixed points are points in a system's phase space where the sum of forces and moments acting at each coordinate is zero. A fixed point is also called an *equilibrium point*. Having information about the structure of the full mass matrix (which includes the Jacobian) at or near fixed points allows us to understand (1) the natural frequencies (eigenvalues) and (2) the distribution in coordinates of each oscillatory mode (eigenvectors). But, if the system is in equilibrium, it will not move; therefore, to get the desired frequency information, the system must be perturbed slightly to unbalance the forces and moments acting at each point. We will call the perturbation δ. The equation of motion for δ is

$$\mathbf{M}\ddot{\delta} + \mathbf{K}\delta = \mathbf{0}. \tag{1}$$

Here, $\mathbf{K}$ represents the stiffness matrix of the system and has the form

$$K_{ij} = \frac{\partial \mathbf{F}_i}{\partial \delta_j}. \tag{2}$$

Here, $\mathbf{F}$ is the total force vector (including reaction forces) on the body.

$\mathbf{M}$ is an invertible matrix, therefore (1) can be reduced to the familiar second order ODE

$$\ddot{\delta} = -\mathbf{M}^{-1}\mathbf{K}\delta \tag{3}$$

which looks a lot like the 1D ODE

$$\ddot{x} = -\omega^2 x,$$

the simple harmonic oscillator! It turns out that the eigenvalues of the matrix $\mathbf{M}^{-1}\mathbf{K}$ are the natural frequencies (squared) of oscillation for the original nonlinear system considered. These frequencies give invaluable information to systems' designers regarding design and control of multibody systems.

The above construction has focused on the separation of reaction forces from the LHS of the original equation of motion presented, however due to practical considerations for this project (the architecture of **dap3d**) it is desirable to keep the original form of the matrix intact. Therefore, we are interested in evaluating the eigenvalues of the matrix $\mathbf{M}_*^{-1}\mathbf{K}_*$, where the matrices are defined as follows

$$\mathbf{M}_* = \begin{pmatrix} \mathbf{M} & -\mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{pmatrix} \mathbf{K}_* = \begin{pmatrix} \mathbf{K_g} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

$\mathbf{K_g}$ is interpreted as the stiffness matrix due only to applied (not reaction) forces. It is currently unknown if using this construction will change the answers from just using pure $\mathbf{M}$ and $\mathbf{K}$ constructions. This project focuses on when each construction should be used.

Examples

1D Spring

The first example to be considered is a bar of length l constrained to 1D motion by two springs (see Figure 1).

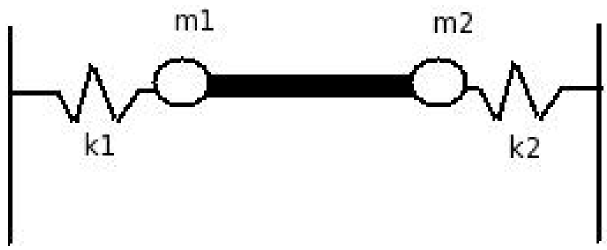

Figure 1: Diagram of 1D spring system

In the case of a system involving springs, the point-to-point stiffness is just the spring constant from Hooke's law ($F = -k\Delta x$). Clearly, this system has only 1 degree of freedom (2 coordinates x_1, x_2 minus the length constraint). From a cursory inspection (and a little knowledge of second-order systems), we know that the natural frequency of this system is

$$\omega = \sqrt{\frac{k1 + k2}{m1 + m2}}.$$

In the multibody formulation, the system's dynamics can be represented by the matrices

$$\mathbf{M}_* = \begin{pmatrix} m_1 & 0 & 1 \\ 0 & m_2 & -1 \\ -1 & 1 & 0 \end{pmatrix} \mathbf{K}_* = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let's simulate the response of this system (and evaluate eigenvalues of above) with the following numerical values (MKS units):

$$\begin{aligned} m_1 &= 3.1 \\ m_2 &= 1.9 \\ k_1 &= 14 \\ k_2 &= 10.7. \end{aligned}$$

From above, $\omega^2 = 4.94(rad/sec)^2$. Using the Matlab command **eig**, the eigenvalues of $\mathbf{M}_*^{-1}\mathbf{K}_*$ are found to be $\{4.94, 0, 0\}$. This shows that the method is working for this system, even with the augmented mass and stiffness matrices. Such may not be the case when applying the method to a nonlinear system.

This result makes sense from an intuitive standpoint; the additional column of zeros to $\mathbf{K}_*$ will add one zero eigenvalue to $\mathbf{M}_*^{-1}\mathbf{K}_*$, and the length constraint will add the other. Therefore, it is expected that a system with n DoF will have n non-zero natural frequencies (eigenvalues).

2D Spring

For the second example, consider the 2D spring problem pictured in Figure 2.

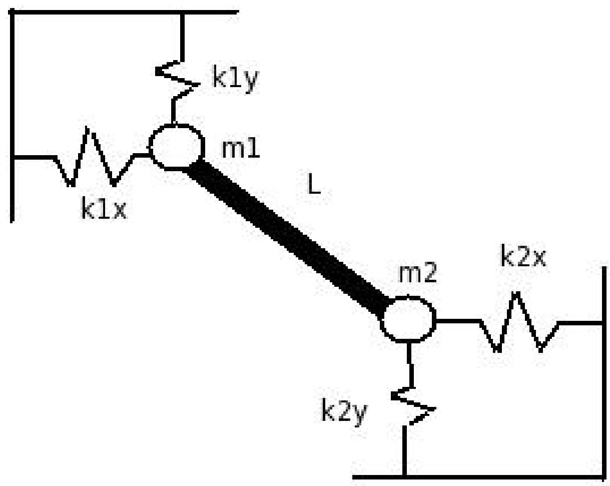

Figure 2: Diagram of 2D spring system

This system has one length constraint and four coordinates, therefore there are three DoF. In our construction, we expect to have 3 non-zero eigenvalues representing the natural frequencies of system. However, due to the power of 2 in the length constraint, this is a nonlinear problem. Therefore, we need to be concerned with what point we choose to linearize about. In this case, it is easiest to linearize about where the springs are not stretched or compressed ($\sum$ Forces $= 0$) and the velocities are zero. For the numerics, let the constants be as follows:

$$\begin{aligned} m_1 &= 1.2 \\ m_2 &= 1.42 \\ k_{1x} &= 204.12 \\ k_{1y} &= 3.77 \\ k_{2x} &= 100000 \\ k_{2y} &= 4000.4 \\ L &= 1. \end{aligned}$$

Putting together the matrices yields

$$\mathbf{M}_* = \begin{pmatrix} m_1 & 0 & 0 & 0 & x_2 - x_1 \\ 0 & m_1 & 0 & 0 & y_2 - y_1 \\ 0 & 0 & m_2 & 0 & x_1 - x_2 \\ 0 & 0 & 0 & m_2 & y_1 - y_2 \\ x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 \end{pmatrix} \mathbf{K}_* = \begin{pmatrix} k_{1x} & 0 & 0 & 0 & 0 \\ 0 & k_{1y} & 0 & 0 & 0 \\ 0 & 0 & k_{2x} & 0 & 0 \\ 0 & 0 & 0 & k_{2y} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\mathbf{M}_*^{-1}\mathbf{K}_*$ are $\{54507.6, 2000.4, 85.5, 0, 0\}$. The largest two eigenvalues are presumably translational modes (the largest in x, due to the large spring constant acting on x_2 displacement from equilibrium). As a check to see if this is correct, we need to simulate the system undergoing translational motion and do a Fourier transform on the time-domain signal of each point's displacement to see how closely the frequencies match.

Before performing the Fourier transform, observe that the first two eigenvalues above correspond to frequencies of 37.2 Hz and 7.1 Hz, respectively. To initialize the systems, the masses were given identical velocities and started from their equilibrium points. The Fourier spectrum is shown in Figure 3.

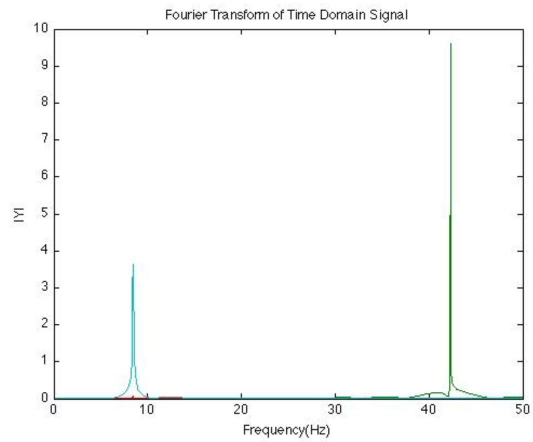

Figure 3: Fourier Transform of 2D Spring System

The frequency spikes in the Fourier spectrum correspond to the dominant oscillatory modes in the system, i.e. the natural frequencies. These spikes occur at 42.3 Hz and 8.5 Hz. These values are close to those predicted using the eigenvalues of $\mathbf{M}_*^{-1}\mathbf{K}_*$, however they err by enough to cause concern. To see if I can do better, I will look at the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$, where

$$\mathbf{K} = \begin{pmatrix} k_{1x} & 0 & k_{2x} & 0 \\ 0 & k_{1y} & 0 & k_{2y} \\ k_{1x} & 0 & k_{2x} & 0 \\ 0 & k_{1y} & 0 & k_{2y} \end{pmatrix}.$$

Note: using this construction, we only get the translational frequencies (squared). In order to get the rotational frequency, we need to add an equation describing the rotational motion about the bar CG. The eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ correspond to frequencies of 42.3 Hz and 8.45 Hz. These values match very closely to what's expected, given the frequency spikes seen in Figure 3.

For this nonlinear system, the construction using $\mathbf{M}_*^{-1}\mathbf{K}_*$ did not work. To be honest, I don't fully understand why and I didn't have enough time to investigate. It seems that all of the necessary information regarding reaction forces should be contained within $\mathbf{M}_*^{-1}\mathbf{K}_*$, however simulation dictates otherwise. This is unfortunate, considering that using $\mathbf{M}_*^{-1}\mathbf{K}_*$ gave a third eigenvalue that was roughly the expected magnitude of the rotational frequency. The verdict for this project is that the full representation $\mathbf{M}_*^{-1}\mathbf{K}_*$ cannot be used to reliably predict eigenvalues and modes for multibody systems; the reduced $\mathbf{M}$ and $\mathbf{K}$ must be used.

Linear Quarter Car

The final example I consider is the linear quarter car suspension model seen Figure 4.

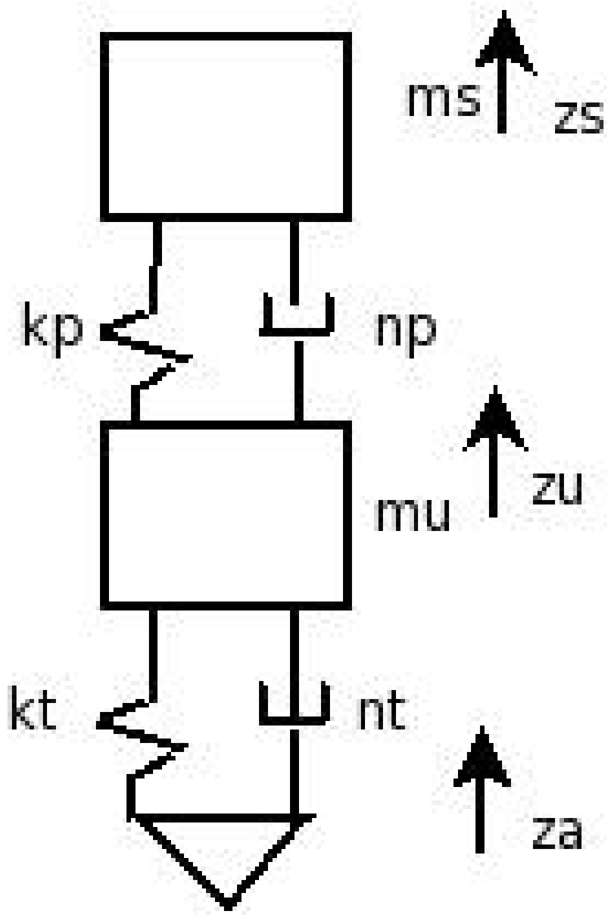

Figure 4: Linear Model of a Quarter Car Suspension System

This system has three coordinates, displayed in Figure 4, one of which (z_a) has a driving constraint. This leaves two degrees of freedom, and therefore two non-zero eigenvalues are expected. To illustrate the concept, I will use **dap3d** to simulate the system response. First, though, I will calculate the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ using the following numerical parameters:

$$\begin{aligned} m_s &= 241.5 \\ m_u &= 41.5 \\ m_a &= 50 \\ k_p &= 140000 \\ k_t &= 400000 \\ n_p &= 6000 \\ n_t &= 10000 \end{aligned}$$

Using these parameters, the natural frequencies are calculated as $\{9638.6, 579.7, 0\}$. These values correspond to 15.6Hz and 3.8Hz, respectively. Now, to verify, I will run a simulation.

Assume the initial conditions are

$$\begin{aligned} z_u &= 0.8 \\ z_s &= 0.5 \\ z_a &= 0 \end{aligned}$$

and that the system is driven by a 10Hz sinusoid f such that

$$\frac{d^2 f}{dt^2} = 20 \sin(2\pi 10 t).$$

This sinusoidal response has been made large to make the Fourier spectrum easier to interpret. Figure 5 shows both the time- and frequency-domain signals for z_s.

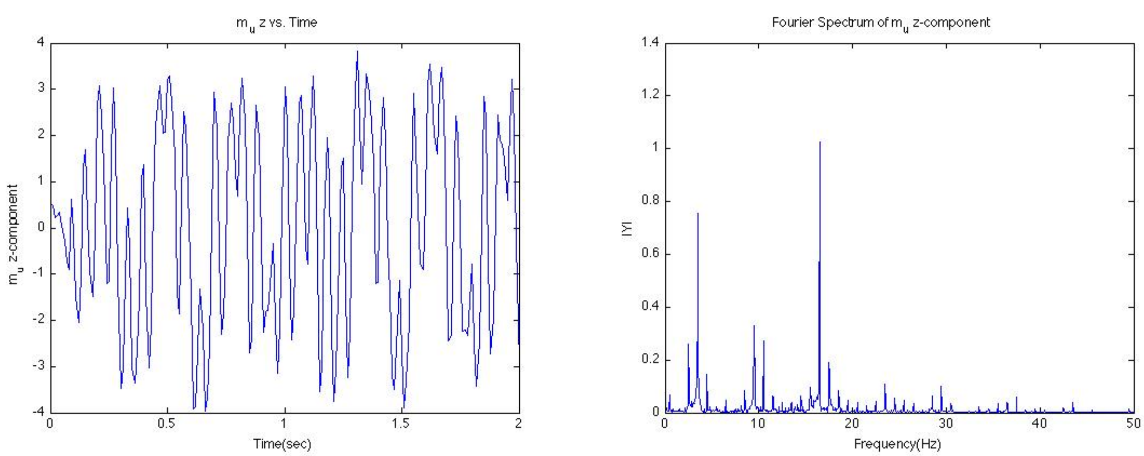

Figure 5: Quarter Car Signals. Left: Time-Domain and Right: Frequency-Domain

The three most prominent spikes in the Fourier spectrum are at 3.5Hz, 9.5Hz, and 16.5Hz. The second value is due to the driving sinusoid, while the first and third values represent the fundamental frequencies of the system. These values are close to expected, given the calculation of the above paragraph.

Conclusion

Methods of eigenanalysis of multibody systems are presented and used in conjunction with simulation to calculate natural frequencies of three examples: a 1D spring, a 2D spring system and a linear quarter car suspension system. In the first example, usage of the full $\mathbf{M}_*^{-1}\mathbf{K}_*$ led to the desired natural frequency calculation. However, this method was shown to give incorrect results in the 2D spring case. It appears as though the reaction forces must be included with the applied forces in the calculation of the stiffness matrix. This makes sense intuitively, as a reaction force is a force acting between two points. For the second and third examples, the reduced matrix $\mathbf{M}^{-1}\mathbf{K}$ was used. This calculation process was verified using methods of discrete Fourier analysis.

References

[1] Andersen, Erik Ryan. *Multibody Dynamics Modeling and System Identification for a Quarter-car Test Rig with McPherson Strut Suspension*. Master's thesis available at *http://scholar.lib.vt.edu/theses/available/etd-05242007-201729/unrestricted*

[2] Nikravesh, Parviz. Lecture Notes for AME553, Lessons 12-15. Available at *www.u.arizona.edu/ pen/ame553/lessons.html*